Well quasi-order for equivalence relations under the consecutive order

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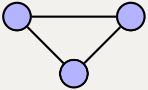
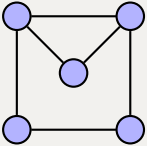
Structures and substructures

Words	aab	$baaba$
Permutations	132	25413
Graphs		

Structures and substructures

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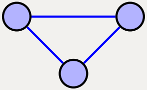
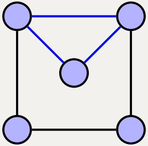
Graphs  

Posets

Two structures of the same kind will be related if and only if one is a substructure of the other; in this way, we form a poset.

Definition

A poset $(X, \leq)$ is a set X together with a reflexive, antisymmetric, transitive binary relation $\leq$ on X. This relation will be called an order.

Examples

1. The set of finite graphs with the subgraph order.
2. The set of finite graphs with the induced subgraph order.
3. The set A^+ with the subword order.

Well quasi-order

Interesting properties of posets include: well quasi-order, atomicity, labelled well quasi-order and better quasi-order.

Definition

An antichain is a set $\{a_1, a_2, \dots\}$ such that $a_i \not\leq a_j$ if $i \neq j$.

Eg. The words $aba, abba, abbba, abbbba, \dots$ form an antichain.

Definition

A poset is well quasi-ordered (wqo) if it contains no infinite antichains (or infinite descending sequences).

Eg. The set of words containing only the letter a is wqo as it forms a chain

$$a \leq aa \leq aaa \leq \dots$$

so there are no antichains at all.

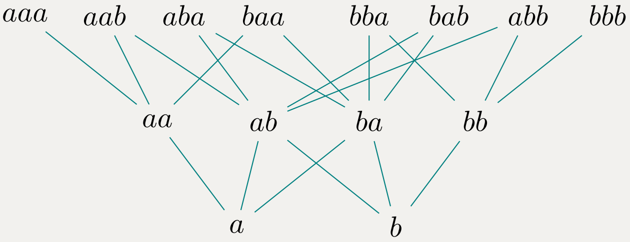
WQO is often taken to be an indicator of the 'wildness' of a poset – those which are wqo are comparatively 'tame'.

Avoidance sets - intuition

We can also ask about properties of subsets of posets, and some subsets of interest are avoidance sets.

We've seen that A^+ with the subword order is not wqo if $|A| > 1$.

What about its subsets?



We can get a subset by chopping off parts of the diagram, eg avoiding aaa, ba as subwords.

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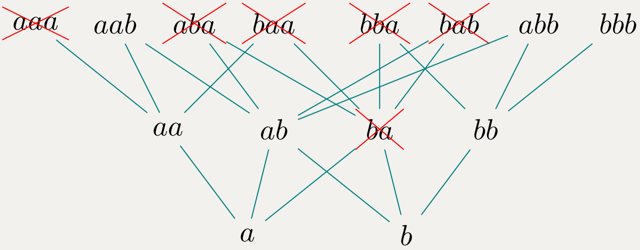


We can get a subset by chopping off parts of the diagram, eg avoiding aaa, ba as subwords.

Avoidance sets - definition and decidability

Definition

If C is a collection of combinatorial structures with an order $\leq$ and $B \subseteq C$ is finite, the avoidance set of B is

$$\mathrm{Av}(B) = \{c \in C \mid \forall b \in B,\ \ b \not\leq c\}.$$

Example

For $\{a, b\}^+$ with the subword order, $aaba \in \mathrm{Av}(aaa)$, but $baaab \notin \mathrm{Av}(aaa)$.

Avoidance sets give rise to natural decidability questions: with input B, we ask about decidability of properties of $\mathrm{Av}(B)$.

The wqo problem

- $(C, \leq)$ - a poset of combinatorial structures

The WQO Problem: Is it decidable, given $B \subseteq C$ finite, whether if $\mathrm{Av}(B)$ is wqo?

Note: if $(C, \leq)$ is wqo, its avoidance sets are also wqo so the wqo problem is trivially decidable.

Overview - results for graphs

Theorem (Ding, 1992)

The wqo problem is decidable for graphs under the subgraph order.

Theorem (Robertson & Seymour, 2004)

The set of all graphs is wqo under the graph minor order (so the wqo problem is trivially decidable).

Open question.

Is the wqo problem decidable for graphs under the induced subgraph order?

Overview - results for permutations

Definition

If σ, ρ are permutations, $\sigma \leq \rho$ under the non-consecutive order iff σ is isomorphic to a subsequence of ρ.

Example

$132 \leq 42513$ as 132 is isomorphic to 253; and $321 \leq 21543$.

Open question.

Is the wqo problem decidable for permutations under the non-consecutive order?

Definition

If σ, ρ are permutations, $\sigma \leq \rho$ under the consecutive order iff σ is isomorphic to a consecutive subsequence of ρ. Eg $321 \leq 21543$ but $132 \not\leq 42513$.

Theorem (McDevitt & Ruškuc, 2021)

The wqo problem is decidable for permutations under the consecutive order.

Overview - results for words

Definition

$u \leq v$ under the non-consecutive order iff u is a non-consecutive subword of v, eg $aa \leq abba$ and $abc \leq babcc$.

Theorem (Higman, 1952)

If A is a finite alphabet, A^+ is wqo under the non-consecutive order.

Definition

$u \leq v$ under the consecutive order iff u is a consecutive subword of v, eg $abc \leq babcc$ but $aa \not\leq abba$.

Lemma

We've seen that A^+ is wqo under the consecutive order iff $|A| = 1$.

Theorem (McDevitt & Ruškuc, 2021)

The wqo problem is decidable for words under the consecutive order.

WQO for structures with consecutive orders

- The wqo problem has been studied for words and permutations under consecutive orders (McDevitt & Ruškuc, 2021).
- This was done by associating avoidance sets with certain digraphs/automata called *factor graphs*.

Structure	Conditions on factor graph for wqo
Words	no in-out cycles
Permutations	no in-out cycles no ambiguous cycles no bicycle has a splittable pair

Can we do the same for equivalence relations?

Equivalence relations

Definition

An equivalence relation on $X = \{1, \ldots, n\}$ is a binary relation on X which is reflexive, symmetric and transitive. It partitions X into equivalence classes.

Examples

$$|1\,3|2|4| \qquad |1\,2\,4|3\,6|5|$$

Isomorphic equivalence relations

Definition

Two equivalence relations are isomorphic if, when we relabel their smallest points 1, second smallest 2, etc, they are identical.

Example

$|1\,6\,|\,3\,| \cong |\,1\,3\,|\,2\,| \cong |\,3\;\;9\,|\,5\,|.$

We consider isomorphic equivalence relations to be equal and let Eq be the set of all equivalence relations on finite sets.

The consecutive order

Definition

σ is a sub-equivalence relation of ρ iff it can be embedded in ρ, i.e. there is a 1-1 map $f : \sigma \to \rho$ such that:

1. If $f(1) = k$ then $f(2) = k + 1$, $f(3) = k + 2, \ldots,$ and
2. x, y are in the same class of $\sigma \iff$ $f(x), f(y)$ are in the same class of ρ.

Then we say $\sigma \leq \rho$ under the consecutive order.

Example

$|\,1\,2\,|\,3\,| \leq |\,1\,2\,3\,|\,4\,5\,|$ as $f : x \mapsto x + 1$ defines a 1-1 map between them satisfying conditions (1) and (2):
$|\,1\,2\,|\,3\,| \leq |\,1\,2\,3\,|\,4\,5\,|$.

But $|\,1\,2\,|\,3\,| \not\leq |\,1\,|\,2\,4\,|\,3\,5\,|$.

Another example

Definition

σ is a sub-equivalence relation of ρ iff it can be embedded in ρ, i.e. there is a 1-1 map $f : \sigma \to \rho$ such that:

1. If $f(1) = k$ then $f(2) = k + 1$, $f(3) = k + 2, \ldots,$ and
2. x, y are in the same class of $\sigma \iff$ $f(x), f(y)$ are in the same class of ρ.

This is written $\sigma \leq \rho$.

Example

$|\,1\,|\,2\,|\,3\,4\,| \leq |\,1\,2\,5\,6\,|\,3\,|\,4\,7\,|$ as the map $f : x \mapsto x + 2$ gives a 1-1 map between them satisfying both conditions.

For condition (2), see that f preserves the equivalence classes:
$|\,1\,|\,2\,|\,3\;\;4\,| \leq |\,1\,2\;5\;\;6\,|\,3\,|\,4\;7\,|$.

The wqo problem for equivalence relations

$\{|1\,n|\,2\ldots n-1|: n\geq 5\}$ is an infinite antichain of equivalence relations.

Why? If $|1\,5|\,2\,3\,4|\leq|1\,6|\,2\,3\,4\,5|$ we would have to map the class $|1\,5|$ to $|1\,6|$, forcing $1\mapsto 1$ and $5\mapsto 6$. But if $1\mapsto 1$, we have $5\mapsto 5$, a contradiction, so $|1\,5|\,2\,3\,4|\not\leq|1\,6|\,2\,3\,4\,5|$.

So $(Eq,\leq)$ is not wqo, and it makes sense to ask about the wqo problem for $(Eq,\leq)$:

The WQO Problem: Is it decidable, given $B\subseteq Eq$ finite, whether $\mathrm{Av}(B)$ is wqo?

We tackle the wqo problem by relating it to similar questions about digraphs.

Next we take a short detour to introduce the necessary ideas from graph theory.

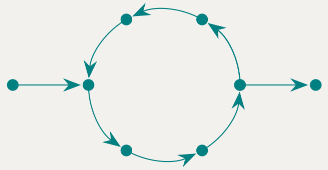
Ideas from graph theory

Definition

If η, π are paths in a finite digraph, then $\eta \leq \pi$ under the subpath order if and only if η is a subpath of π.

Definition

A cycle in a digraph is an in-out cycle if at least one vertex has in degree > 1 and at least one vertex has out degree > 1.



Theorem (McDevitt & Ruškuc, 2021)

The set of paths of a finite digraph G under the subpath order is wqo if and only if G contains no in-out cycles.

Factor graphs

Returning to equivalence relations, let $B \subset Eq$ and consider $\mathrm{Av}(B)$.

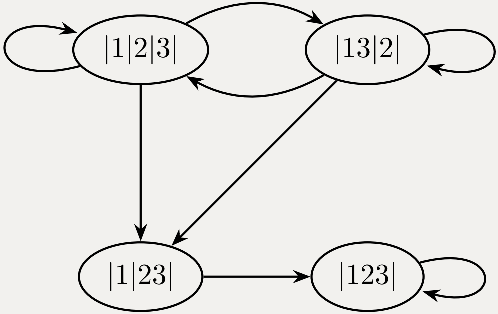
Let b be the maximum length of an equivalence relation in B.

The factor graph of B is the digraph Γ_B with:

- Vertices: equivalence relations of length b in $\mathrm{Av}(B)$.
- Edges: $\sigma \to \tau$ iff the last $b-1$ points of σ are isomorphic to the first $b-1$ points of τ, or formally, $\sigma\restriction_{[2,b]} \cong \tau\restriction_{[1,b-1]}$.

Factor graph example

The factor graph of $\{|\,1\,2\,|\,3\,|\}$ is:



$$|\,1\,|\,2\,|\,3\,| \to |\,1\,3\,|\,2\,| \text{ because } |\,2\,|\,3\,| \cong |\,1\,|\,2\,|$$

Factor graph example

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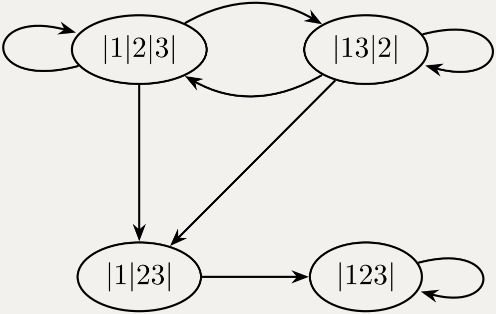
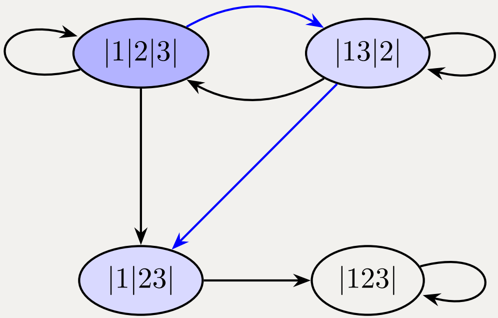


$|\,1\,3\,|\,2\,| \to |\,1\,3\,|\,2\,|$ because $|\,3\,|\,2\,| \cong |\,1\,|\,2\,|$.

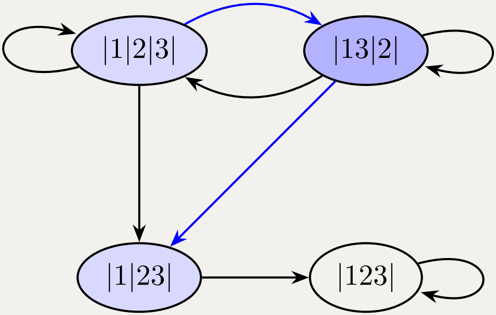
Equivalence relations trace paths



Example

$\sigma = |\,1\,|\,2\;4\;5\,|\,3\,|$ traces the path
$\Pi(\sigma) = |\,1\,|\,2\,|\,3\,| \to |\,1\;3\,|\,2\,| \to |\,1\,|\,2\;3\,|$.

Equivalence relations trace paths



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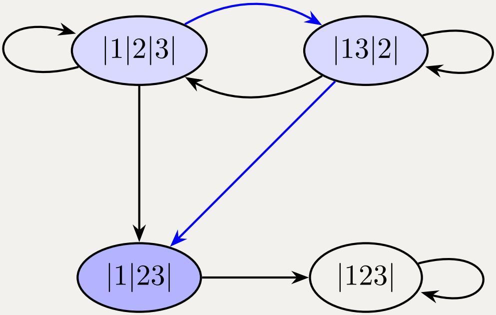
Equivalence relations trace paths



Example

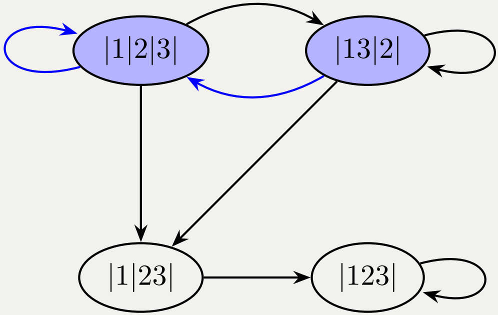
$\sigma = |\,1\,|\,2\;4\;5\,|\,3\,|$ traces the path

$\Pi(\sigma) = |\,1\,|\,2\,|\,3\,| \to |\,1\,3\,|\,2\,| \to |\,1\,|\,2\;3\,|.$

Sub-equivalence relations and subpaths

Lemma: If $\sigma \leq \rho$, then $\Pi(\sigma) \leq \Pi(\rho)$.

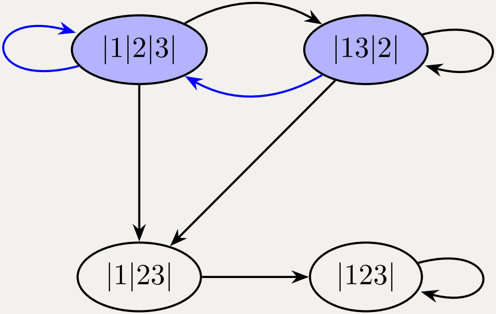
Paths can be traced by > 1 equivalence relation



Example

$|\,1\ \ 3\,|\,2\,| \to |\,1\,|\,2\,|\,3\,| \to |\,1\,|\,2\,|\,3\,|$ is traced by $|\,1\ \ 3\,|\,2\,|\,4\,|\,5\,|$ and $|\,1\ \ 3\,|\,2\ \ 5\,|\,4\,|$.

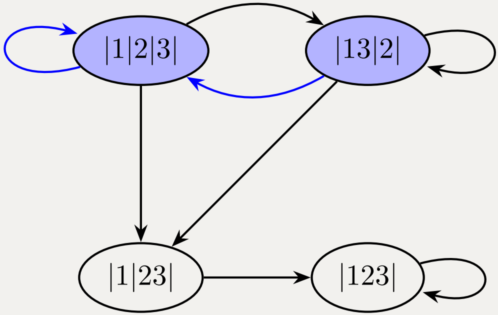
Paths can be traced by > 1 equivalence relation



Example

$|1\,3|2| \to |\,\boxed{1}\,|\,\boxed{2}\,|\,\boxed{3}\,| \to |1|2|3|$ is traced by $|1\,\boxed{3}\,|\,\boxed{2}\,|\,\boxed{4}\,|5|$ and $|1\,\boxed{3}\,|\,\boxed{2}\,5|\,\boxed{4}\,|$.

Paths can be traced by > 1 equivalence relation



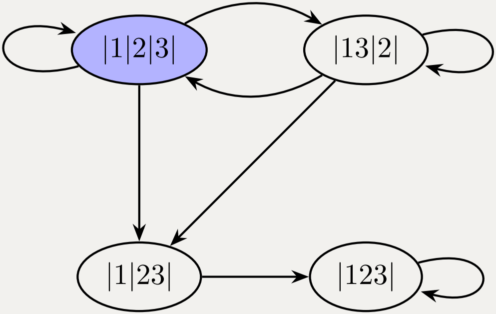
Example

$|1\,3|\,2| \rightarrow |1|\,2|\,3| \rightarrow |\,1\,|\,2\,|\,3\,|$ is traced by $|1\,3\,|2|\,4\,|\,5\,|$ and $|1\,3\,|2\,5\,|\,4\,|$.

Special vertices

Definition

A special vertex is one where the largest entry is in a class of size one.



Special vertices

Definition

A special vertex is one where the largest entry is in a class of size one.

Special vertices can give choices in the placement of the next entry of an equivalence relation - it can either be added to:

1. a brand new class; or
2. an existing class containing much smaller elements.

Example

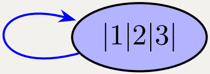
$|\,1\,3\,|\,2\,| \to |\,1\,|\,2\,|\,3\,| \to |\,1\,|\,2\,|\,3\,|$ was traced by

1. $|\,1\,3\,|\,2\,|\,4\,|\,5\,|$, where 5 was added to a new class; and
2. $|\,1\,3\,|\,2\,5\,|\,4\,|$, where 5 was added to the class of 2.

Special vertices in cycles

Lemma

If the factor graph Γ_B *contains a special vertex in a cycle,* $\mathrm{Av}(B)$ *is not wqo.*



Consider the equivalence relations:

- Which trace paths that go around the cycle $i \geq 3$ times;
- Where a new entry is added to an existing class the first and last times we enter the special vertex;
- Where a new class is created whenever else we enter the special vertex.

$|1\,4|2\,6|3|5|, \ |1\,4|2\,7|3|5|6|, \ |1\,4|2\,8|3|5|6|7|, \ldots$.
These form an infinite antichain.

In this way, an infinite antichain can be created from any cycle containing a special vertex.

Antichains from in-out cycles

Recall:

1. a digraph contains an in-out cycle iff its paths are not wqo (McDevitt & Ruškuc, 2021);
2. if $\Pi(\sigma) \not\leq \Pi(\rho)$ then $\sigma \not\leq \rho$.

Lemma
If Γ_B contains an in-out cycle, $\mathrm{Av}(B)$ *is not wqo.*

Proof.

- Γ_B is not wqo so contains an infinite antichain of paths $\pi_1, \pi_2, \ldots$ by (1).
- Take $\sigma_1, \sigma_2, \ldots$ s.t. σ_i traces π_i.
- By (2), $\sigma_1, \sigma_2, \ldots$ is an infinite antichain of equivalence relations.

□

Decidability result

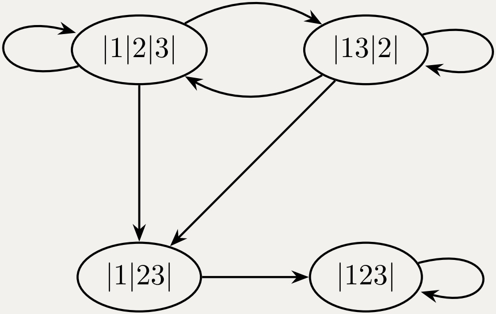
Theorem: $\mathrm{Av}(B)$ is wqo if and only if the factor graph Γ_B contains no in-out cycles or special vertices in cycles.
(VI & Ruškuc, 2023)

Theorem: It is decidable whether $\mathrm{Av}(B)$ is wqo, so the wqo problem is decidable for equivalence relations under the consecutive order. (VI & Ruškuc, 2023)

Example

Example

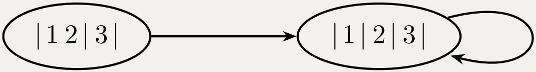
Av(| 1 2 | 3 |) is not wqo as its factor graph contains both in-out cycles and a special vertex in a cycle:



More examples

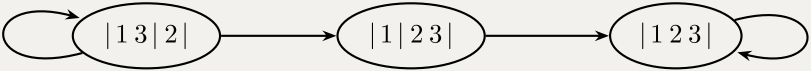
Example

$\mathrm{Av}(|1\,2\,3|, |1\,3\,|\,2|, |1\,|\,2\,3|)$ is not wqo as its factor graph contains a special vertex in a cycle:



Example

$\mathrm{Av}(|1\,|\,2\,|\,3|, |1\,2\,|\,3|)$ is wqo as its factor graph has neither in-out cycles nor special vertices in cycles:



Comparisons

Structure	Conditions on factor graph for wqo
Words	no in-out cycles
Equivalence relations	no in-out cycles no special vertices in cycles
Permutations	no in-out cycles no ambiguous cycles no bicycle has a splittable pair

Further questions - other structures

- We ask the wqo problem for other combinatorial structures under consecutive orders.
- Ongoing work is looking at this for structures consisting of several equivalence relations and permutations.
- It would also be interesting to investigate the wqo problem for digraphs.

Further questions - varying the order

Definition

Equivalence relations σ, ρ are related under the non-consecutive order iff there is an 1-1 map $f : \sigma \to \rho$ such that

- x, y are in the same class of $\sigma \Leftrightarrow f(x), f(y)$ are in the same class of ρ.

Lemma (VI & Ruškuc, 2023)

The poset of equivalence relations under the non-consecutive order is wqo, so the wqo problem is trivially decidable.

Theorem

The wqo problem is decidable for equivalence relations under the consecutive order.

Question: If we change the order to respect the underlying linear order, but not consecutively, can we answer the wqo problem?

Further questions - atomicity

Definition

If $(X, \leq)$ is a poset and $C \subseteq X$:

- C is downward closed if $c \in C$ and $a \leq c$ imply $a \in C$.
- If C downward closed, C is atomic if it cannot be expressed as a union of two downward closed, proper subsets.

Theorem

A downward closed subset C of $(X, \leq)$ is atomic if and only if it satisfies the joint embedding property: for any $x, y \in C$ there exists $z \in C$ such that $x, y \leq z$. (Fraïssé, 1954)

Examples

- $(\mathbb{N}, \leq)$ is atomic.
- $(\{1, \ldots, 10\}, \leq_d)$, where $\leq_d$ is the divisibility order, is not atomic as no numbers are divisible by 3 and 5.

Further questions - atomicity

Structure	Conditions on factor graph for atomicity
Words	strongly connected or a bicycle small words well behaved
Equivalence relations	strongly connected or a bicycle with no ambiguous vertices small relations well behaved
Permutations	strongly connected or a bicycle with no ambiguous cycles small permutations well behaved

(Words & permutations by McDevitt & Ruškuc, 2021.)

Questions:

- Can we answer the atomicity problem for other structures under consecutive orders?
- Is there an overarching picture behind these results?

Thank you for listening!